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THE ANNULAR CRACK SURROUNDING AN ELASTIC FIBER IN A TENSION FIELD

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Abstract—Matrix cracking in brittle-matrix fiber-reinforced composites is investigated when the fracture strain of the fiber is greater than that of the matrix. The axisymmetric problem of an infinitely long elastic fiber perfectly bonded to an elastic matrix which contains an annular crack surrounding the fiber is considered for the case of uniform longitudinal strain. The problem is formulated in terms of a singular integral equation with a Cauchy type kernel. When the inner crack tip terminates at the interface, it is shown that the characteristic equation is the same as that for the case of plane strain. Stress intensity factors at the crack tips are given when (a) the inner crack tip is away from the interface and (b) the inner crack tip is at the interface.

INTRODUCTION

The present investigation concerns brittle-matrix fiber-reinforced composites subjected to longitudinal tension. Prior to the first crack appearing in the undamaged composite, the longitudinal tensile strain is uniform and the same in the fiber and matrix. In general the fracture strain of the fiber is much higher than that of the matrix and the initial cracks that appear in the matrix are in a plane perpendicular to the direction of loading. The fracture process could initiate at the site of imperfections such as impurities or voids in the matrix. When the applied load is increased, the matrix cracks propagate and surround the fibers. Redistribution of load takes place between the fibers and the matrix, with the fibers carrying more load than the matrix.

The pioneering work on matrix cracking in brittle-matrix fiber-reinforced composites is due to the efforts of Aveston *et al.* (1971) who introduced certain concepts which are now known as the ACK theory. More recently ACK theory has been further extended and improved upon by Marshall *et al.* (1985), Budiansky *et al.* (1986) and McCartney (1987).

In this paper the axisymmetric problem of an infinitely long fiber embedded in an infinite matrix with an annular crack surrounding the fiber is considered. (See Fig. 1.) The case of a single fiber is taken to represent the extreme case where the spacing between fibers is large compared to the radius of the fiber. The problem is formulated such that any axisymmetric longitudinal tensile loading can be considered.

The problem of an annular crack located in an isotropic homogeneous elastic solid has been considered previously by Choi and Shield (1982), Nied and Erdogan (1983), Selvadurai and Singh (1985) and by Clements and Ang (1988). Earlier work on this problem was carried out by Smetanin (1968), Moss and Kobayashi (1971) and Shibuya *et al.* (1975).

FORMULATION OF THE PROBLEM

An infinitely long elastic fiber of radius a is perfectly bonded to the elastic matrix which contains an annular crack surrounding the fiber, as shown in Fig. 1. The inner and outer radii of the crack are b and c, respectively ($b < c < \infty$). A uniform longitudinal tensile strain ε_0 is applied to the system at $z = \pm\infty$ and the matrix is not constrained at $r = \infty$. The required solution is obtained by the superposition of the solutions of two related problems. In the first problem the perfectly bonded fiber and matrix in the absence of the

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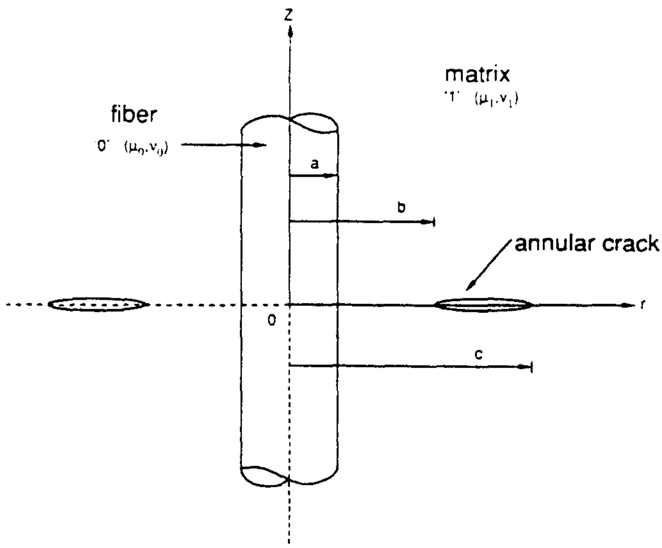


Fig. 1. An annular crack in the matrix surrounding an elastic fiber.

annular crack are subjected to the uniform longitudinal tensile strain ε_0, while the matrix is allowed to deform freely at $r = \infty$. The first problem can be solved without much difficulty and the stress fields are given in Appendix A. The matrix stresses required for the second problem are $\sigma^1_{zz}(r, 0) = E_1\varepsilon_0$ and $\sigma^1_{rz}(r, 0) = 0$ where E_1 is the Young's modulus of the matrix. Since $\sigma^1_{rz}(r, 0)$ is identically equal to zero, the stresses applied to the crack surfaces in the second problem are those equal and opposite to the stresses $\sigma^1_{zz}(r, 0)$. In this paper we consider the problem where the crack surface tractions are the only external loads.

For axially symmetric problems the non-vanishing displacements and stresses can be expressed in terms of Love's stress function $\chi(r, z)$ as follows (Love, 1944, p. 276; Timoshenko and Goodier, 1970, p. 381)

$$u_r(r, z) = -\frac{1}{2\mu}\frac{\partial^2\chi}{\partial r\partial z}, \tag{1}$$

$$u_z(r, z) = \frac{1}{2\mu}\left[2(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right], \tag{2}$$

$$\sigma_{rr}(r, z) = \frac{\partial}{\partial z}\left[\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial r^2}\right], \tag{3}$$

$$\sigma_{\theta\theta}(r, z) = \frac{\partial}{\partial z}\left[\nu\nabla^2\chi - \frac{1}{r}\frac{\partial\chi}{\partial r}\right], \tag{4}$$

$$\sigma_{zz}(r, z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right], \tag{5}$$

$$\sigma_{rz}(r, z) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right], \tag{6}$$

where $\chi(r, z)$ is an axisymmetric biharmonic function, ∇^2 is the axisymmetric Laplacian, μ is the shear modulus and ν is Poisson's ratio. Since $z = 0$ is a plane of symmetry the semi-infinite domain $z \geq 0$ is considered.

The fiber $\chi^0(r, z)$ is defined by

$$\chi^0(r,z) = \frac{2}{\pi}\int_0^\infty [f_1(s)I_0(rs)+f_2(s)rsI_1(rs)]\sin(zs)\,ds + \int_0^\infty f_3(p)p(2\nu_0+zp)\,e^{-zp}J_0(rp)\,dp, \quad (7)$$

where the functions f_i ($i = 1, 2, 3$) are yet to be determined, $J_n(\ )$ is the Bessel function of the first kind of order n and $I_n(\ )$ is the modified Bessel functions of the first kind of order n. Hence the displacements and relevant stress components in the fiber can be expressed as

$$u_r^0(r,z) = -\frac{1}{2\mu_0}\left\{\frac{2}{\pi}\int_0^\infty [f_1(s)I_1(rs)+f_2(s)rsI_0(rs)]s^2\cos(zs)\,ds - \int_0^\infty f_3(p)p^3(1-2\nu_0-zp)\,e^{-zp}J_1(rp)\,dp\right\}, \quad (8)$$

$$u_z^0(r,z) = \frac{1}{2\mu_0}\left\{\frac{2}{\pi}\int_0^\infty [f_1(s)I_0(rs)+f_2(s)[4(1-\nu_0)I_0(rs)+rsI_1(rs)]]s^2\sin(zs)\,ds - \int_0^\infty f_3(p)p^3[2(1-\nu_0)+zp]\,e^{-zp}J_0(rp)\,dp\right\}, \quad (9)$$

$$\sigma_{rr}^0(r,z) = \frac{2}{\pi}\int_0^\infty \{f_1(s)[-I_0(rs)+I_1(rs)/rs]+f_2(s)[(2\nu_0-1)I_0(rs)-rsI_1(rs)]\} \times s^2\cos(zs)\,ds + \int_0^\infty f_3(p)p^4[(1-zp)J_0(rp)-(1-2\nu_0-zp)J_1(rp)/rp]\,e^{-zp}\,dp, \quad (10)$$

$$\sigma_{zz}^0(r,z) = \frac{2}{\pi}\int_0^\infty \{f_1(s)I_0(rs)+f_2(s)[2(2-\nu_0)I_0(rs)+rsI_1(rs)]\}s^3\cos(zs)\,ds + \int_0^\infty f_3(p)p^4(1+zp)\,e^{-zp}J_0(rp)\,dp, \quad (11)$$

$$\sigma_{rz}^0(r,z) = \frac{2}{\pi}\int_0^\infty \{f_1(s)I_1(rs)+f_2(s)[rsI_0(rs)+2(1-\nu_0)I_1(rs)]\}s^3\sin(zs)\,ds + \int_0^\infty f_3(p)p^5z\,e^{-zp}J_1(rp)\,dp, \quad (12)$$

where μ_0 and ν_0 are the fiber shear modulus and Poisson's ratio, respectively.

The matrix $\chi^1(r,z)$ is defined by

$$\chi^1(r,z) = \frac{2}{\pi}\int_0^\infty [f_4(s)K_0(rs)+f_5(s)rsK_1(rs)]\sin(zs)\,ds + \int_0^\infty f_6(p)p(2\nu_1+zp)\,e^{-zp}J_0(rp)\,dp, \quad (13)$$

where the functions f_i ($i = 4, 5, 6$) are still to be determined and $K_n(\ )$ is the modified Bessel function of the second kind of order n. The displacements and relevant stress components in the matrix are expressed as

$$u_r^1(r,z) = \frac{1}{2\mu_1}\left\{\frac{2}{\pi}\int_0^\infty [f_4(s)K_1(rs)+f_5(s)rsK_0(rs)]s^2\cos(zs)\,\mathrm{d}s + \int_0^\infty f_6(p)p^3(1-2\nu_1-zp)\,\mathrm{e}^{-zp}J_1(rp)\,\mathrm{d}p\right\}, \quad (14)$$

$$u_z^1(r,z) = \frac{1}{2\mu_1}\left\{\frac{2}{\pi}\int_0^\infty [f_4(s)K_0(rs)+f_5(s)[-4(1-\nu_1)K_0(rs)+rsK_1(rs)]]s^2\sin(zs)\,\mathrm{d}s - \int_0^\infty f_6(p)p^3[2(1-\nu_1)+zp]\,\mathrm{e}^{-zp}J_0(rp)\,\mathrm{d}p\right\}, \quad (15)$$

$$\sigma_{rr}^1(r,z) = \frac{2}{\pi}\int_0^\infty \{-f_4(s)[K_0(rs)+K_1(rs)/rs]+f_5(s)[(1-2\nu_1)K_0(rs)-rsK_1(rs)]\} \times s^3\cos(zs)\,\mathrm{d}s + \int_0^\infty f_6(p)p^4[(1-zp)J_0(rp)-(1-2\nu_1-zp)J_1(rp)/rp]\,\mathrm{e}^{-zp}\,\mathrm{d}p, \quad (16)$$

$$\sigma_{zz}^1(r,z) = \frac{2}{\pi}\int_0^\infty \{f_4(s)K_0(rs)+f_5(s)[-2(2-\nu_1)K_0(rs)+rsK_1(rs)]\}s^3\cos(zs)\,\mathrm{d}s + \int_0^\infty f_6(p)p^4(1+zp)\,\mathrm{e}^{-zp}J_0(rp)\,\mathrm{d}p, \quad (17)$$

$$\sigma_{rz}^1(r,z) = \frac{2}{\pi}\int_0^\infty \{-f_4(s)K_1(rs)+f_5(s)[-rsK_0(rs)+2(1-\nu_1)K_1(rs)]\}s^3\sin(zs)\,\mathrm{d}s + \int_0^\infty f_6(p)p^5z\,\mathrm{e}^{-zp}J_1(rp)\,\mathrm{d}p \quad (18)$$

where μ_1 and ν_1 are the matrix shear modulus and Poisson's ratio, respectively.

The boundary conditions at the interface are

$$u_r^0(a,z) = u_r^1(a,z), \quad u_z^0(a,z) = u_z^1(a,z), \quad 0 \leqslant z < \infty, \quad (19)$$

$$\sigma_{rr}^0(a,z) = \sigma_{rr}^1(a,z), \quad \sigma_{rr}^0(a,z) = \sigma_{rz}^1(a,z), \quad 0 \leqslant z < \infty, \quad (20)$$

while the plane $z = 0$ is subjected to the conditions

$$\sigma_{rz}^0(r,0) = 0, \quad 0 \leqslant r \leqslant a, \quad \sigma_{rz}^1(r,0) = 0, \quad a \leqslant r < \infty, \quad (21)$$

$$\sigma_{zz}^1(r,0) = -p(r), \quad b < r < c, \quad (22)$$

$$u_z^0(r,0) = 0, \quad 0 \leqslant r \leqslant a, \quad u_z^1(r,0) = 0, \quad a \leqslant r < b, \quad c < r < \infty. \quad (23)$$

From eqns (12) and (18) it is seen that eqn (21) is automatically satisfied, while $f_3(p) = 0$ from the first eqn of (23). At this stage a new unknown function $\phi(r)$ is introduced in the cracked region as follows

$$\frac{\mu_1}{1-\nu_1}\frac{\partial}{\partial r}u_z^1(r,0) = \phi(r), \quad b < r < c. \quad (24)$$

Hence from eqns (15), (24) and the second eqn of (23)

$$p^3 f_6(p) = \int_b^c t\phi(t) J_1(pt)\,\mathrm{d}t. \tag{25}$$

The four boundary conditions at the interface, eqns (19) and (20), yield the following four eqns for the unknown functions f_i ($i = 1, 2, 4, 5$) in terms of the yet undetermined function $\phi(r)$ as follows:

$$\bar{\mu} I_1(as) f_1(s) + \bar{\mu} as I_0(as) f_2(s) + K_1(as) f_4(s) + as K_0(as) f_5(s) = \frac{1}{s^3}\int_b^c t\phi(t) h_1(t,s)\,\mathrm{d}t, \tag{26}$$

$$-\bar{\mu} I_0(as) f_1(s) - \bar{\mu}[4(1-\nu_0) I_0(as) + as I_1(as)] f_2(s) + K_0(as) f_4(s)$$
$$+[-4(1-\nu_1) K_0(as) + as K_1(as)] f_5(s) = \frac{1}{s^3}\int_b^c t\phi(t) h_2(t,s)\,\mathrm{d}t, \tag{27}$$

$$[I_0(as) - I_1(as)/as] f_1(s) + [-(2\nu_0 - 1) I_0(as) + as I_1(as)] f_2(s)$$
$$-[K_0(as) + K_1(as)/as] f_4(s) + [(1-2\nu_1) K_0(as)$$
$$-as K_1(as)] f_5(s) = \frac{1}{s^3}\int_b^c t\phi(t) h_3(t,s)\,\mathrm{d}t, \tag{28}$$

$$-I_1(as) f_1(s) - [as I_0(as) + 2(1-\nu_0) I_1(as)] f_2(s) - K_1(as) f_4(s)$$
$$+[-as K_0(as) + 2(1-\nu_1) K_1(as)] f_5(s) = \frac{1}{s^3}\int_b^c t\phi(t) h_4(t,s)\,\mathrm{d}t, \tag{29}$$

where $\bar{\mu} = \mu_1/\mu_0$ and the functions h_i ($i = 1, \ldots, 4$) are given by

$$h_1(t,s) = -s\{-as I_0(as) K_1(ts) + ts I_1(as) K_0(ts) + 2(1-\nu_1) I_1(as) K_1(ts)\}, \tag{30}$$

$$h_2(t,s) = -s\{-ts I_0(as) K_0(ts) + 2(1-\nu_1) I_0(as) K_1(ts) + as I_1(as) K_1(ts)\}, \tag{31}$$

$$h_3(t,s) = -s\left\{ts I_0(as) K_0(ts) + I_0(as) K_1(ts) - \frac{t}{a} I_1(as) K_0(ts)\right.$$
$$\left. - \left[as + \frac{2(1-\nu_1)}{as}\right] I_1(as) K_1(ts)\right\}, \tag{32}$$

$$h_4(t,s) = -s\{as I_0(as) K_1(ts) - ts I_1(as) K_0(ts)\}. \tag{33}$$

In deriving eqns (26)–(29) and later on in this section, certain infinite integrals are evaluated by making use of the formulae found in Erdelyi (1954). Solving eqns (26)–(29), f_i ($i = 1, 2, 4, 5$) can be expressed as

$$f_1(s) = \int_b^c t\phi(t)\,\mathrm{d}t \frac{1}{s^3} \sum_{i=1}^{4} \frac{A_i(s) h_i(t,s)}{\Delta(s)}, \tag{34}$$

$$f_2(s) = \int_b^c t\phi(t)\,\mathrm{d}t \frac{1}{s^3} \sum_{i=1}^{4} \frac{B_i(s) h_i(t,s)}{\Delta(s)}, \tag{35}$$

$$f_4(s) = \int_b^c t\phi(t)\,\mathrm{d}t \frac{1}{s^3} \sum_{i=1}^{4} \frac{C_i(s) h_i(t,s)}{\Delta(s)}, \tag{36}$$

$$f_5(s) = \int_b^c t\phi(t)\,dt \frac{1}{s^3}\sum_{i=1}^{4} \frac{D_i(s)h_i(t,s)}{\Delta(s)}, \tag{37}$$

where $\Delta(s)$ is the determinant and A_i, B_i, C_i, D_i, $(i = 1,\ldots,4)$ are the appropriate elements of the adjoint of the coefficient matrix of eqns (26)–(29). From eqns (22) and (17), after substituting for f_i $(i = 4, 5, 6)$ from eqns (36), (37) and (25) the following integral equation is obtained:

$$\frac{1}{\pi}\int_b^c \left[\frac{1}{t-r} + k(r,t)\right]\phi(t)\,dt = -p(r), \quad b < r < c \tag{38}$$

where

$$k(r,t) = k_1(r,t) + 2tk_2(r,t),$$

$$k_1(r,t) = \frac{m(r,t)-1}{t-r} + \frac{m(r,t)}{t+r}, \tag{39}$$

$$m(r,t) = \begin{cases} E(r/t), & r < t \\ \dfrac{r}{t}E(t/r) + \dfrac{(t^2-r^2)}{rt}K(t/r), & r > t \end{cases} \tag{40}$$

$$k_2(r,t) = \int_0^\infty \bar{k}_2(r,t,s)\,ds, \tag{41}$$

$$\bar{k}_2(r,t,s) = \frac{1}{\Delta(s)}\left\{\left(\sum_{i=1}^{4} C_ih_i\right)K_0(rs) + \left(\sum_{i=1}^{4} D_ih_i\right)[-2(2-\nu_1)K_0(rs) + rsK_1(rs)]\right\}, \tag{42}$$

where $K(\ )$ and $E(\ )$ are the complete elliptic integrals of the first and second kind, respectively. The singular integral equation of the first kind eqn (38) is solved under the crack closure condition

$$\int_b^c \phi(r)\,dr = 0, \tag{43}$$

which is obtained from the second eqn of (23) and eqn (24). The physical significance of eqn (43) is that the crack tips are closed at b and c.

INNER CRACK TIP AWAY FROM INTERFACE

When the inner crack tip is located away from the interface, i.e. $b > a$ the dominant kernel in the integral eqn (38) is the term $1/(t-r)$. The kernel $k_1(r,t)$ has a logarithmic singularity of the form $\log|t-r|$ and $k_2(r,t)$ is bounded in the interval $b \leqslant r, t \leqslant c$. The solution of eqn (38) is of the form

$$\phi(t) = [(t-b)(c-t)]^{-1/2}g_1(t), \quad b < t < c \tag{44}$$

where $g_1(t)$ is a bounded function. Normalizing the interval (b, c) by defining

$$t = \frac{c-b}{2}\tau + \frac{c+b}{2}, \quad r = \frac{c-b}{2}\rho + \frac{c+b}{2}, \tag{45}$$

$$\phi(t) = h_1(\tau) = F_1(\tau)(1-\tau^2)^{-1/2}, \tag{46}$$

$$p(r) = P(\rho), \quad K(\rho, \tau) = \frac{c-b}{2}k(r, t), \tag{47}$$

we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\tau-\rho} + K(\rho, \tau)\right]\frac{F_1(\tau)}{(1-\tau^2)^{1/2}}\,\mathrm{d}\tau = -P(\rho), \quad -1 < \rho < 1 \tag{48}$$

$$\int_{-1}^{1}\frac{F_1(\tau)}{(1-\tau^2)^{1/2}}\,\mathrm{d}\tau = 0. \tag{49}$$

The function $F_1(\tau)$ is obtained by using a Gauss–Chebyshev type quadrature formula (Erdogan and Gupta, 1972) and solving the singular integral eqn (48) numerically under the constraint condition (49).

The stress intensity factors defined by

$$K(b) = \lim_{r \to b}\sqrt{2(b-r)}\sigma_{zz}^1(r, 0), \tag{50}$$

$$K(c) = \lim_{r \to c}\sqrt{2(r-c)}\sigma_{zz}^1(r, 0), \tag{51}$$

can be expressed as

$$K(b) = \lim_{r \to b}\sqrt{2(r-b)}\phi(r) = a_1^{1/2}F_1(-1), \tag{52}$$

$$K(c) = -\lim_{r \to c}\sqrt{2(c-r)}\phi(r) = -a_1^{1/2}F_1(1), \tag{53}$$

where $a_1 = (c-b)/2$. Since $F_1(\tau)$ is determined at discrete points away from the end points -1 and $+1$, recourse is made to the formulae given by Krenk (1975) to obtain $F_1(-1)$ and $F_1(1)$.

INNER CRACK TIP AT INTERFACE

For the case $b = a$, i.e. when the inner crack tip is at the interface, $k_2(r, t)$ given by eqn (41) is no longer bounded for all r, t in the closed interval $[b, c]$. By adding and subtracting the asymptotic value of $\bar{K}_2(r, t, s)$ for large values of s, from the integrand in eqn (41), $k_2(r, t)$ may be expressed as

$$k_2(r, t) = k_{2f}(r, t) + k_{2s}(r, t), \tag{54}$$

where

$$k_{2f}(r, t) = \int_0^{\infty}[\bar{K}_2(r, t, s) - \bar{K}_2^{\infty}(r, t, s)]\,\mathrm{d}s + \Gamma_{2f}^{c}(r, t), \tag{55}$$

$$\bar{K}_2^{\infty}(r, t, s) = -[R_1 s^2 + R_2 s + R_3]\frac{e^{-s(r+t-2a)}}{2\sqrt{rt}}, \tag{56}$$

$$\Gamma_{2f}^{\varepsilon}(r,t) = \frac{1}{2\sqrt{rt}}\left[\sum_{i=1}^{6} S_i + \sum_{i=1}^{3} T_i\right], \tag{57}$$

$$k_{2s}(r,t) = \Gamma_{2s}^{\varepsilon}(r,t), \tag{58}$$

$$\Gamma_{2s}^{\varepsilon}(r,t) = \frac{1}{2\sqrt{rt}}\left\{\frac{c_0}{(r+t-2a)} - \frac{c_1(r-a)}{(r+t-2a)^2} + \frac{2c_2(r-a)^2}{(r+t-2a)^3}\right\}, \tag{59}$$

$$c_0 = \frac{1}{2}\left[1 - \frac{(1+\kappa_1)}{1+\bar{\mu}\kappa_0} + \frac{3(1-\bar{\mu})}{\bar{\mu}+\kappa_1}\right], \quad c_1 = \frac{6(1-\bar{\mu})}{\bar{\mu}+\kappa_1}, \quad c_2 = \frac{2(1-\bar{\mu})}{\bar{\mu}+\kappa_1}, \tag{60}$$

where R_i, T_i ($i = 1, 2, 3$) and S_i ($i = 1, \ldots, 6$) are given in Appendix B and $\kappa_i = 3-4\nu_i$ ($i = 1, 2$). In eqn (55), $k_{2f}(r, t)$ is bounded in $[b, c]$ for all r, t and $k_{2s}(r, t)$ is unbounded as r and t approach a. The singular kernel $k_{2s}(r, t)$ is of the same form as that given by Erdogan *et al.* (1973) for the corresponding plane strain problem.

Equations (38) can now be rewritten as

$$\frac{1}{\pi}\int_a^c \frac{\phi(t)}{t-r}\,\mathrm{d}t + \frac{1}{\pi}\int_a^c \mathrm{l}_1(r,t)\phi(t)\,\mathrm{d}t + \frac{1}{\pi}\int_a^c \mathrm{l}_2(r,t)\phi(t)\,\mathrm{d}t = -p(r), \quad a<r<c \tag{61}$$

where

$$\mathrm{l}_1(r,t) = 2tk_{2s}(r,t), \tag{62}$$

$$\mathrm{l}_2(r,t) = k_1(r,t) + 2tk_{2f}(r,t), \tag{63}$$

and where $\mathrm{l}_2(r, t)$ is a Fredholm kernel.

The solution to eqn (61) is expressed as

$$\phi(t) = (c-t)^{\alpha}(t-a)^{\beta}g_2(t), \tag{64}$$

and following the technique given by Muskhelishvili (1953, Chapter 5) which is also elaborated by Erdogan *et al.* (1973) the following characteristic equations are obtained to determine α and β:

$$\cot \pi\alpha = 0, \tag{65}$$

$$2d_1 \cos \pi(\beta+1) - d_2(\beta+1)^2 - d_3 = 0, \tag{66}$$

where

$$d_1 = (1+\bar{\mu}\kappa_0)(\bar{\mu}+\kappa_1), \tag{67}$$

$$d_2 = 4(1-\bar{\mu})(1+\bar{\mu}\kappa_0), \tag{68}$$

$$d_3 = -(1-\bar{\mu})(1+\bar{\mu}\kappa_0) + (1+\bar{\mu}\kappa_0)(\bar{\mu}+\kappa_1) - (1+\kappa_1)(\bar{\mu}+\kappa_1). \tag{69}$$

It is noted that eqns (65) and (66) are identical to that obtained in the plane strain case. Equation (65) yields $\alpha = -\frac{1}{2}$ which is the well-known singularity for a crack tip surrounded by a homogeneous medium. The real constant β is a function of the material properties of the fiber and matrix.

Normalizing the interval (a, c) by defining

$$t = \frac{c-a}{2}\tau + \frac{c+a}{2}, \quad r = \frac{c-a}{2}\rho + \frac{c+a}{2}, \tag{70}$$

$$\phi(t) = h_2(\tau) = F_2(\tau)(1-\tau)^{\alpha}(\tau+1)^{\beta}, \tag{71}$$

$$L_1(\rho,\tau) = \frac{c-a}{2} l_1(r,t), \quad L_2(\rho,\tau) = \frac{c-a}{2} l_2(r,t), \tag{72}$$

$$p(r) = P(\rho), \tag{73}$$

we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{1}{\tau-\rho} + L_1(\rho,\tau) + L_2(\rho,\tau)\right\} F_2(\tau)(1-\tau)^{\alpha}(\tau+1)^{\beta}\,d\tau = -P(\rho), \quad -1 < \rho < 1. \tag{74}$$

The crack closure condition (43) yields the equation

$$\int_{-1}^{1} F_2(\tau)(1-\tau)^{\alpha}(\tau+1)^{\beta}\,d\tau = 0. \tag{75}$$

Equation (74) which is a singular integral equation with a generalized Cauchy kernel is solved numerically by means of Gauss–Jacobi integration formulae (Erdogan *et al.*, 1973), under the crack closure condition eqn (75).

The stress intensity factors are defined by

$$K(c) = \lim_{r\to c} \sqrt{2(r-c)}\,\sigma^{1}_{zz}(r,0), \tag{76}$$

$$K(a) = \lim_{r\to a} 2^{1/2}(a-r)^{-\beta}\sigma^{0}_{zz}(r,0). \tag{77}$$

Making use of the fact that the left-hand side of eqn (38) yields an expression for $\sigma^{1}_{zz}(r,0)$, for $r > c$, it can be shown that

$$K(c) = -2^{1/2}(c-a)^{\beta}g_2(c) = -\lim_{r\to c} 2^{1/2}(c-r)^{-\alpha}\phi(r) = -2^{\beta+1/2}a_2^{1/2}F_2(1), \tag{78}$$

where $a_2 = (c-a)/2$.

$\sigma^{0}_{zz}(r,0)$ is obtained from eqns (11), (34) and (35) and the following expression is obtained for $K(a)$

$$K(a) = 2^{1/2}\mu^*(c-a)^{\alpha}g_2(a) - \mu^*\lim_{r\to a} 2^{1/2}(r-a)^{-\beta}\phi(r) = \mu^* a_2^{-\beta}F_2(-1), \tag{79}$$

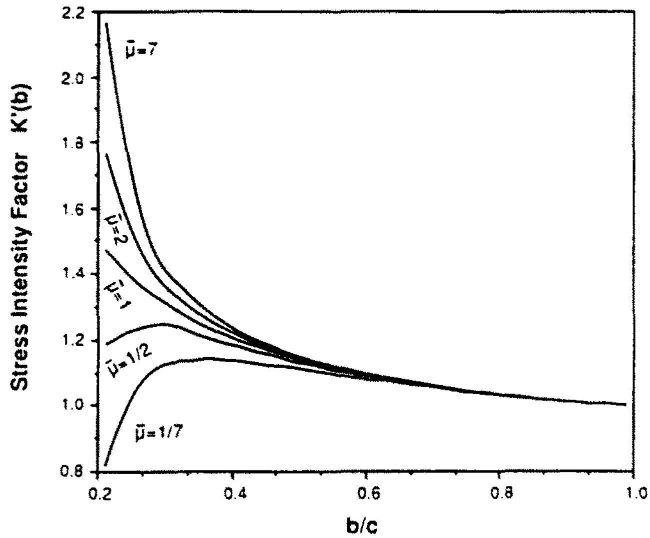
where

$$\mu^* = -\frac{1+\kappa_1}{2}\left\{\frac{(3+2\beta)(\bar{\mu}+\kappa_1)-(1+2\beta)(1+\bar{\mu}\kappa_0)}{(1+\bar{\mu}\kappa_0)(\bar{\mu}+\kappa_1)\sin\pi(1+\beta)}\right\}. \tag{80}$$

Quadratic extrapolation was used to obtain $F_2(-1)$ and $F_2(1)$.

RESULTS AND DISCUSSION

In the numerical examples the Poisson's ratios were taken as $\nu_0 = \nu_1 = 0.25$ and $p(r) = E_1\varepsilon_0$ [eqn (A6), Appendix A]. The limiting case of $c \to \infty$, which results in an external



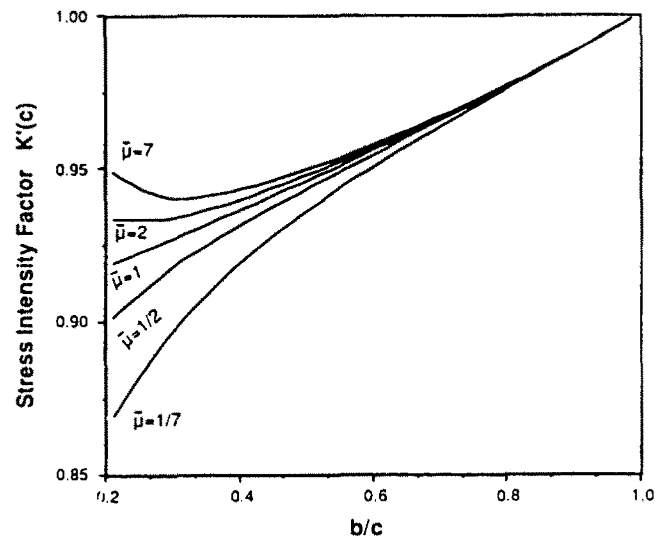


Fig. 2. Stress intensity factors when the inner crack tip is away from the interface, $a/c = 0.2$, $\nu_0 = \nu_1 = 0.25$.

crack with uniform loading cannot be considered using this formulation. When the inner crack tip is away from the interface, i.e. when both the crack tips have square-root singularities, the normalized stress intensity factors are defined by

$$K'(b) = \frac{K(b)}{\sigma_0 a_1^{1/2}} = F_1(-1), \quad K'(c) = \frac{K(c)}{\sigma_0 a_1^{1/2}} = -F_1(1), \tag{81}$$

where $\sigma_0 = E_1\varepsilon_0$. Figures 2 and 3 show the normalized stress intensity factors for the ratios $a/c = 0.2$ and 0.5, respectively. When the size of the crack is very small, i.e. when $b/c \to 1.0$, the stress intensity factors are not sensitive to the presence of the fiber neither are they influenced by the curvature of the body for all values of $\bar{\mu}$; $K'(b)$ and $K'(c) \to 1.0$ which is the result for the case of a Griffith crack in a homogeneous, isotropic matrix in plane strain. When $\bar{\mu} > 1$, $K'(b) > K'(c)$ which implies that the crack would propagate inward towards the center. When $\mu_0 \gg \mu_1$ and $b \to a$, $K'(b) < K'(c)$ due to the presence of the fiber. For a

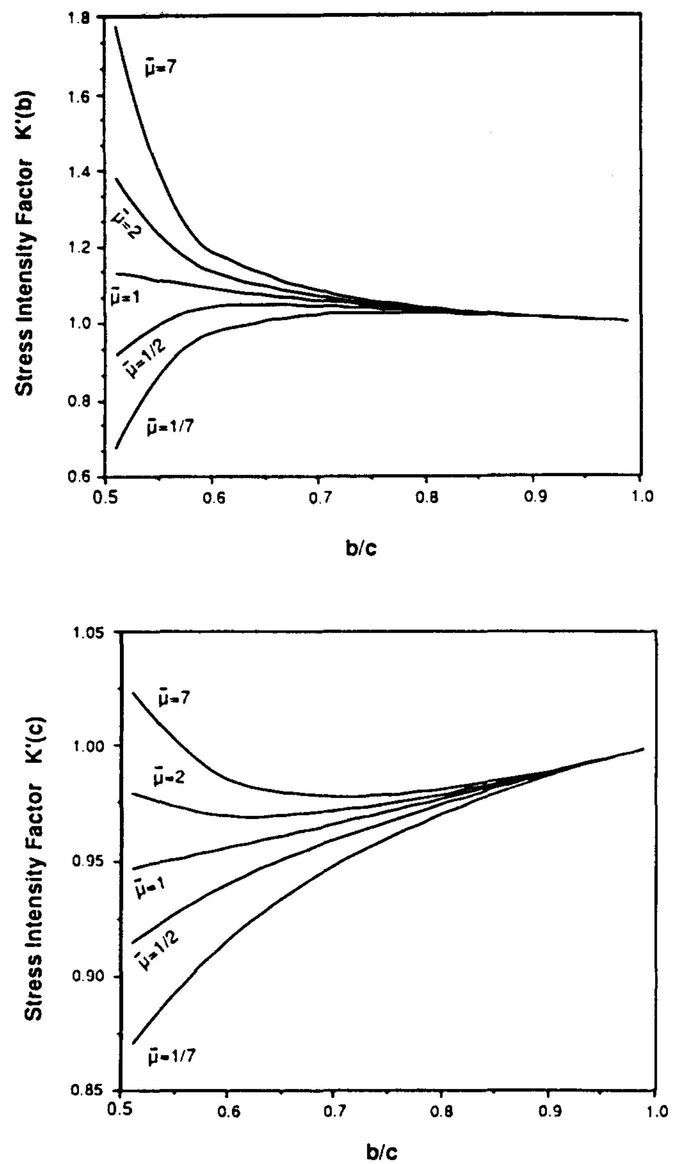


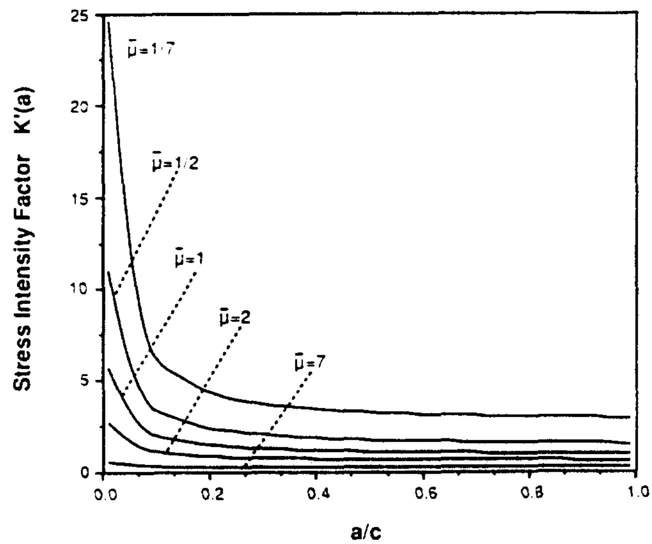
Fig. 3. Stress intensity factors when the inner crack tip is away from the interface, $a/c = 0.5$, $\nu_0 = \nu_1 = 0.25$.

given crack size, i.e. for fixed b/c, both $K'(b)$ and $K'(c)$ decrease with decreasing $\bar{\mu}$, as expected.

When the inner crack tip is at the interface the normalized stress intensity factors are defined by

$$K'(a) = \frac{K(a)}{\sigma_0 a_2^{\beta}} = \mu^* F_2(-1), \quad K'(c) = \frac{K(c)}{\sigma_0 a_2^{1/2}} = -2^{\beta+1/2} F_2(1). \tag{82}$$

For $\bar{\mu} = 1/7$, $1/2$, 1.0, 2.0 and 7.0, β takes the values -0.3304, -0.4295, -0.5, -0.5755 and -0.7149, respectively. The normalized stress intensity factors are given in Fig. 4. $K'(a)$ increases with decreasing $\bar{\mu}$, but the inner crack tip singularity decreases with decreasing $\bar{\mu}$ as expected and hence it is not possible to compare $K'(a)$ for different ratios of $\bar{\mu}$. For a given value of $\bar{\mu}$, when the position of the outer crack tip is held fixed $K'(a)$ increases as the radius of the fiber gets smaller. Since the outer crack tip singularity is independent of $\bar{\mu}$, $K'(c)$ decreases with decreasing $\bar{\mu}$ as expected. When $a/c \to 0$, where c is finite and $a \to$



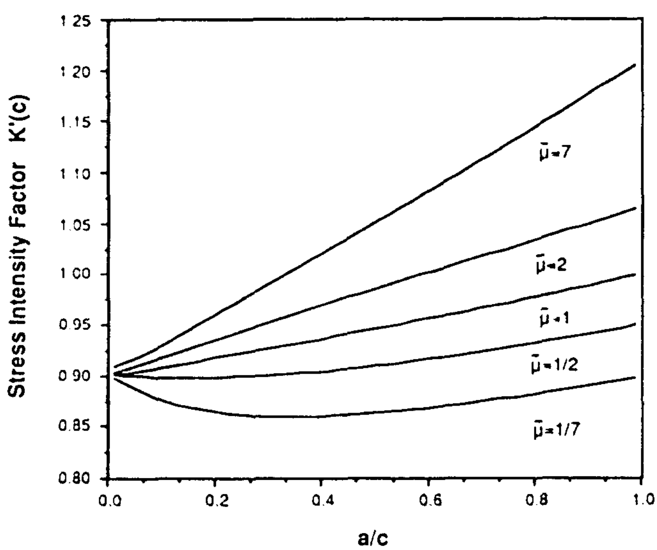


Fig. 4. Stress intensity factors when the inner crack tip is at the interface, $\nu_0 = \nu_1 = 0.25$.

0, $K'(c) \to 2\sqrt{2}/\pi$, the solution for the penny-shaped crack in a homogeneous, isotropic matrix. For $\bar{\mu} = 1.0$ only, $K'(c) \to 1.0$ as $a/c \to 1.0$ since $K'(c)$ is dependent on β as shown in eqn (82). Both curves $K'(a)$ and $K'(c)$ for the case $\bar{\mu} = 1.0$ agree with the results of Clements and Ang (1988).

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APPENDIX A

Stress fields when the fiber and matrix are subjected to a uniform longitudinal tensile strain ε_0 at $z = \pm\infty$ and the matrix is unconstrained at $r = \infty$ (Chawla, 1987, p. 188) are

$$\sigma^0_{rr}(r) = \sigma^* \tag{A1}$$

$$\sigma^1_{rr}(r) = \frac{a^2}{r^2}\sigma^* \tag{A2}$$

$$\sigma^0_{\theta\theta}(r) = \sigma^* \tag{A3}$$

$$\sigma^1_{\theta\theta}(r) = -\frac{a^2}{r^2}\sigma^* \tag{A4}$$

$$\sigma^0_{zz}(r) = E_0\varepsilon_0 + 2\nu_0\sigma^* \tag{A5}$$

$$\sigma^1_{zz}(r) = E_1\varepsilon_0 \tag{A6}$$

$$\sigma^0_{rz}(r) = 0 \tag{A7}$$

$$\sigma^1_{rz}(r) = 0 \tag{A8}$$

where

$$\sigma^* = \frac{2\varepsilon_0\mu_0\mu_1(\nu_0-\nu_1)}{\mu_1(1-2\nu_0)+\mu_0} \tag{A9}$$

and μ, ν and E are the shear modulus, Poisson's ratio and Young's modulus respectively. The superscripts and subscripts 0 and 1 refer to the fiber and matrix, respectively.

APPENDIX B

The functions R_i ($i = 1, 2, 3$) appearing in eqn (56) are given by,

$$R_1 = \frac{P_1}{Q_1}, \tag{B1}$$

$$R_2 = \frac{1}{Q_1}\left(-\frac{P_1Q_2}{Q_1} + P_2\right), \tag{B2}$$

$$R_3 = \frac{1}{Q_1}\left[P_1\left(-\frac{Q_3}{Q_1} + \frac{Q_2^2}{Q_1^2}\right) - \frac{P_2 Q_2}{Q_1} + P_3\right], \tag{B3}$$

where

$$P_1 = 2(1-\bar{\mu})(1+\bar{\mu}\kappa_0)(r-a)(t-a), \tag{B4}$$

$$P_2 = (1-\bar{\mu})\{2(r-a)(1+\bar{\mu}\kappa_0) - 3(r+t-2a)(1+\bar{\mu}\kappa_0) + (r-a)(r+t-2a)p_{21} - (r-a)^2 p_{21}\}, \tag{B5}$$

$$P_3 = 6 + 4v_1(2v_1 - 3) + \bar{\mu}[-2\bar{\mu}\kappa_0 + 4v_1(1-2v_0)] + (1-\bar{\mu})\left[(r-a)p_{31} + (r+t-2a)p_{32} + \frac{1}{a^2}(r-a)(r+t-2a)p_{33} + (r-a)^2 p_{34}\right], \tag{B6}$$

$$p_{21} = (1+\bar{\mu}\kappa_0)\left(\frac{3}{4t} - \frac{1}{4r}\right) - [5\bar{\mu} + 7 - 4v_0(\bar{\mu}+2)]/2a, \tag{B7}$$

$$p_{31} = (1+\bar{\mu}\kappa_0)\left(\frac{15a}{64t^2} + \frac{9a}{64r^2} + \frac{1}{32r} + \frac{39}{32t}\right) + (93+87\bar{\mu})/64t - (31\bar{\mu}-11)/64r - v_0(21\bar{\mu}+24)/16t + v_0(13\bar{\mu}-8)/16r - 3(5\bar{\mu}+7)/4a + 3v_0(\bar{\mu}+2)/a, \tag{B8}$$

$$p_{32} = -\tfrac{9}{8}(1+\bar{\mu}\kappa_0)\left(\frac{a}{8r^2} + \frac{1}{t}\right) + (31\bar{\mu}-11)/64r - v_0(13\bar{\mu}-8)/16r + (-19\bar{\mu}+63)/16a + v_0(15\bar{\mu}-26)/4a + 2v_1(1+\bar{\mu}\kappa_0)/a, \tag{B9}$$

$$p_{33} = \tfrac{1}{16}[19 - 7\bar{\mu} + 4v_0(\bar{\mu}-4)], \tag{B10}$$

$$p_{34} = -\frac{3}{32rt}(1+\bar{\mu}\kappa_0) - \frac{p_{33}}{a^2}, \tag{B11}$$

$$Q_1 = (1+\bar{\mu}\kappa_0)(\bar{\mu}+\kappa_1), \tag{B12}$$

$$Q_2 = -\frac{2}{a}(1-\bar{\mu}^2)[2 - 3(v_0+v_1) + 4v_0v_1], \tag{B13}$$

$$Q_3 = (1-\bar{\mu})[\tfrac{3}{2}(1-\bar{\mu}) + v_0(5\bar{\mu}-2) + v_1(2\bar{\mu}-5) + 4v_0v_1(1-\bar{\mu})]/a^2. \tag{B14}$$

The functions s_i ($i = 1, \ldots, 6$) and T_i ($i = 1, 2, 3$) required to define $I^s_{ij}(r, t)$ in eqn (57) are expressed as

$$S_1 = \frac{P_1 Q_2}{Q_1^2}\frac{1}{(r+t-2a)^2}, \tag{B15}$$

$$S_2 = -\frac{(r-a)(1-\bar{\mu})}{Q_1(r+t-2a)}p_{21}, \tag{B16}$$

$$S_3 = \frac{(r-a)^2(1-\bar{\mu})}{Q_1(r+t-2a)^2}p_{21}, \tag{B17}$$

$$S_4 = \frac{P_1 Q_2}{Q_1^2}\frac{1}{(r+t-2a)}, \tag{B18}$$

$$S_5 = -\frac{1}{Q_1}(1-\bar{\mu})p_{32}, \tag{B19}$$

$$S_6 = -\frac{(r-a)(1-\bar{\mu})}{Q_1(r+t-2a)}p_{33}, \tag{B20}$$

$$T_1 = -\frac{P_1}{Q_1}\left(-\frac{Q_3}{Q_1} + \frac{Q_2^2}{Q_1^2}\right)\frac{1}{(r+t-2a)}, \tag{B21}$$

$$T_2 = -\frac{1}{a^2 Q_1}(r-a)(1-\bar{\mu})p_{33}, \tag{B22}$$

$$T_3 = -\frac{(r-a)^2(1-\bar{\mu})}{Q_1(r+t-2a)}p_{34}. \tag{B23}$$